Linear Algebra

1. (a) Suppose that $\mathbf{Q}_1$ and $\mathbf{Q}_2$ are two $n \times n$ orthogonal matrices. Show that the product $\mathbf{Q}_1\mathbf{Q}_2$ is also an orthogonal matrix.

   (b) Consider the matrix

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & 2 \\ 1 & 1 & -1 \\ 6 & 0 & 2 \end{bmatrix}.$$

   Determine the nullspace of $\mathbf{A}$.

2. Consider the system of equations

$$\begin{aligned} x_1 - x_2 + x_3 + 2x_4 &= 1, \\ -x_1 + 2x_2 + x_3 - x_4 &= 0, \\ 2x_1 - x_2 - x_3 + 2x_4 &= 1, \\ 11x_1 + x_2 + x_3 - x_4 &= 2, \\ 3x_1 + x_2 + 4x_3 + 5x_4 &= 2. \end{aligned}$$

   Write the above in the form $\mathbf{A}\mathbf{x} = \mathbf{b}$. Using the Gaussian elimination technique to the Augmented matrix $[\mathbf{A}|\mathbf{b}]$, deetermine the rank of the matrix $\mathbf{A}$. Is there a solution to $\mathbf{x}$? If the answer is yes, obtain the solution. If the answer is no, explain.

Linear Algebra

1. (a) Consider an $n \times n$ elementary unit lower-triangular matrix $\mathbf{L}_i$. (Note: A unit lower triangular matrix is a lower triangular matrix with each of the diagonal elements equal to unity. An elementary lower triangular matrix $\mathbf{L}_i$ is a lower triangular matrix with possible non-zero elements below the diagonal element in the i^{th} column.)

   Let $\mathbf{E}_{ij}$ denote the permutation matrix obtained by interchanging the rows i and j of the identity matrix $\mathbf{I}$ of size n.

   Show that the product $\mathbf{E}_{35}\mathbf{L}_1\mathbf{E}_{35}^T$ interchanges only the third and fifth elements in the first column of $\mathbf{L}_1$.

   (b) Show that permutation matrices are orthogonal.

2. (a) A real symmetric matrix $\mathbf{A}$ is said to be positive definite if $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$ ($\mathbf{x}$ is an n-dimensional vector). Consider the following matrix obtained during the discretization of a differential equation:

$$\mathbf{A} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & -1 & \ddots & \ddots & \\ & & \ddots & \ddots & -1 \\ & & & -1 & 2 \end{bmatrix}$$

   Show that $\mathbf{A}$ is positive definite.

   (b) Show that the diagonal elements of a positive definite matrix are positive.

Complex variables

1. Let $u(x,y)$ and $v(x,y)$ be real-valued functions of the real variables x and y.

   Also let $z = x + iy$ and let $w = u(x,y) + iv(x,y)$.

   a) Sketch a straight line going from -1 - i to 2 + i on an Argand diagram. This straight line is represented by the variable z.

   b) Sketch the corresponding line for the complex variable $w = u(x,y) + iv(x,y)$ on an Argand diagram for the transformation $w = 3iz + 1$

   c) As for b), sketch on an Argand diagram the corresponding line for the transformation $w = \frac{1}{z}$

2. With w and z defined in question 1,

   a) State the Cauchy Riemann condition for $u(x,y)$ and $v(x,y)$ to be conjugate functions so that w is an analytic function of z.

   a) A real-valued function of two real variables is said to be harmonic if it satisfies the two-dimensional version of Laplace's equation. Is $v(x,y) = e^x \sin(y)$ harmonic? If so, find the conjugate harmonic function $u(x,y)$ and express w directly as a function of z

   b) Is $u = x^2 - 2y$ harmonic? If so, find the conjugate harmonic function $v(x,y)$ and express w directly as a function of z

Complex variables

In general let $w = u + iv$ and $z = x + iy$ where $i = \sqrt{-1}$.

1. Given that the above complex numbers can be represented on an Argand diagram, sketch the three straight line paths traced out in the z plane starting at the origin, A, proceeding to point B at 1, then to point C located at i before closing at point A. Show these three path segments and use arrows to indicate path directions.

   State the equations for determining points in x and y for each of these three path segments.

   Determine and sketch the corresponding path in the w plane where
   $w = z^2$

2. If $w = \sinh(z)$ show that u and v satisfy the Cauchy-Riemann condition for differentiability as well as the Laplace equation.

   Derive a general expression for $\ln(z)$ and hence determine solutions of

   a) $\ln(-4)$.

   b) $\ln(3-4i)$.