Calculus of One Variable

Summary

This exam addresses fundamental topics from single-variable calculus, including the definitions of continuity and differentiability and the explicit calculation of limits, derivatives, and integrals. In order to earn full credit for your response to a question on this exam, your conceptual assertions must be justified and your calculations shown explicitly.

Sample questions and answers

1. Compute $\displaystyle\int \frac{2x^2+x+18}{x^3+9x}dx$.

   The integrand can be decomposed in the form

$$\frac{2x^2+x+18}{x^3+9x} = \frac{2x^2+x+18}{x(x^2+9)} = \frac{A}{x} + \frac{Bx+C}{x^2+9}.$$

Since

$$\frac{A}{x} + \frac{Bx+C}{x^2+9} = \frac{A(x^2+9)+(Bx+C)x}{x(x^2+9)} = \frac{(A+B)x^2+Cx+9A}{x(x^2+9)},$$

it follows that

$$A+B=2, \qquad C=1, \qquad 9A=18$$

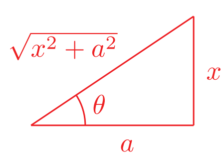
so $A=2$, $B=0$, and

$$\int \frac{2x^2+x+18}{x^3+9x}dx = \int \left(\frac{2}{x}+\frac{1}{x^2+9}\right)dx = \int \frac{2}{x}\,dx + \int \frac{1}{x^2+9}\,dx = 2\ln x + \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right).$$

Note that while it's good to remember the formula

$$\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right),$$

this can be derived using trigonometric substitution. Consider the triangle shown:



Since $\dfrac{x}{a} = \tan\theta$, it follows that

$$d\left(\frac{x}{a}\right) = \frac{dx}{a} = d(\tan\theta) = \frac{d\theta}{\cos^2\theta}.$$

Since $\dfrac{a}{\sqrt{x^2+a^2}} = \cos\theta$, it follows that

$$\frac{1}{x^2+a^2}\,dx = \left(\frac{\cos\theta}{a}\right)^2\left(\frac{a\,d\theta}{\cos^2\theta}\right) = \frac{d\theta}{a}.$$

Thus

$$\int_b^c \frac{dx}{x^2+a^2} = \frac{1}{a}\int_{\tan^{-1}(b/a)}^{\tan^{-1}(c/a)} d\theta = \frac{1}{a}\tan^{-1}\left(\frac{c}{a}\right) - \frac{1}{a}\tan^{-1}\left(\frac{b}{a}\right).$$

2. Consider the function

$$g(x) = \begin{cases} x^2 & \text{if } x \text{ is rational} \\ 0 & \text{if } x \text{ is irrational} \end{cases}.$$

Is $g(x)$ continuous at $x = 0$? Is $g(x)$ differentiable at $x = 0$?

Note that $0 \leq g(x) \leq x^2$ for all values of x. Since $\lim_{x\to 0} 0 = \lim_{x\to 0} x^2 = 0$, it follows (from the "squeeze theorem") that $\lim_{x\to 0} g(x) = 0$. Since $g(0)$ is defined and equal to $\lim_{x\to 0} g(x)$, $g(x)$ is continuous at $x = 0$.

When $x = 0$, the derivative

$$\frac{dg}{dx} = \lim_{\epsilon\to 0} \frac{g(x+\epsilon) - g(x)}{\epsilon}$$

is given by

$$\left.\frac{dg}{dx}\right|_{x=0} = \lim_{\epsilon\to 0} \frac{g(0+\epsilon) - g(0)}{\epsilon} = \lim_{\epsilon\to 0} \frac{g(\epsilon)}{\epsilon}.$$

It follows that $g(x)$ is differentiable at $x = 0$ if this limit exists. Note that

$$\frac{g(\epsilon)}{\epsilon} = \begin{cases} \epsilon & \text{if } \epsilon \text{ is rational} \\ 0 & \text{if } \epsilon \text{ is irrational} \end{cases}$$

as long as $\epsilon \neq 0$, so that

$$-|\epsilon| \leq \frac{g(\epsilon)}{\epsilon} \leq |\epsilon|$$

as long as $\epsilon \neq 0$. Since $\lim_{\epsilon\to 0}(-|\epsilon|) = \lim_{\epsilon\to 0}(|\epsilon|) = 0$, it follows (again from the "squeeze theorem") that $\lim_{\epsilon\to 0} \frac{g(\epsilon)}{\epsilon} = 0$, so $g(x)$ is differentiable at $x = 0$.

Note that if a function is differentiable at a certain point, it must be continuous at this point. Had we shown that $g(x)$ was differentiable at $x = 0$ in the first place, we could have concluded that $g(x)$ was continuous at $x = 0$ without the first proof above.

3. Compute $\lim_{x\to e} \left(\frac{1-\ln x}{x-e}\right)^2$.

Since

$$\lim_{x\to e} \left(\frac{1-\ln x}{x-e}\right)^2 = \lim_{x\to e} \frac{(1-\ln x)^2}{(x-e)^2}$$

and $\lim_{x\to e}(1-\ln x)^2 = \lim_{x\to e}(x-e)^2 = 0$, l'Hôpital's rule may apply. We compute

$$\frac{d}{dx}\left((1-\ln x)^2\right) = 2(1-\ln x)\left(-\frac{1}{x}\right) = \frac{2(\ln x - 1)}{x}$$

and

$$\frac{d}{dx}\left((x-e)^2\right) = 2(x-e).$$

Since both of these quantities tend to 0 as x tends to e, we compute

$$\frac{d}{dx}\left(\frac{2(\ln x - 1)}{x}\right) = \frac{4 - 2\ln x}{x^2}$$

and

$$\frac{d}{dx}\left(2(x-e)\right) = 2.$$

Both of these quantities tend to finite nonzero values as x tends to e, so l'Hôpital's rule does apply and

$$\lim_{x\to e} \left(\frac{1-\ln x}{x-e}\right)^2 = \lim_{x\to e} \frac{\frac{2(\ln x-1)}{x}}{2(x-e)} = \lim_{x\to e} \frac{\frac{4-2\ln x}{x^2}}{2} = \frac{1}{e^2}.$$

4. In order to compute

$$\lim_{x\to\infty} \frac{\pi x + \sin x}{x + \sin x},$$

we try two different methods. First, we perform the calculation

$$\lim_{x\to\infty} \frac{\pi x + \sin x}{x + \sin x} = \lim_{x\to\infty} \frac{\pi + \frac{\sin x}{x}}{1 + \frac{\sin x}{x}} = \pi.$$

Next, to verify this result, we use l'Hôpital's rule, noting that both the numerator and the denominator of the original fraction are differentiable and approach ∞ as $x \to \infty$. We obtain

$$\lim_{x\to\infty} \frac{\pi x + \sin x}{x + \sin x} = \lim_{x\to\infty} \frac{\frac{d}{dx}(\pi x + \sin x)}{\frac{d}{dx}(x + \sin x)} = \lim_{x\to\infty} \frac{\pi + \cos x}{1 + \cos x},$$

which is undefined because $\cos x$ oscillates persistently as $x \to \infty$. We conclude that π is undefined (which isn't true). What went wrong?

L'Hôpital's rule

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

can only be applied to the situation when $\lim_{x\to\infty} f(x) = \pm\infty$ and $\lim_{x\to\infty} g(x) = \pm\infty$ if

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

is defined or is equal to $\pm\infty$. Neither of these criteria is satisfied in the present case.

5. Recall the formula

$$\int u\, dv = uv - \int v\, du$$

for integration by parts. If we apply this formula with $u = 1/x$ and $v = x$, we obtain

$$\int \left(\frac{1}{x}\right) dx = \left(\frac{1}{x}\right) x - \int x \left(-\frac{1}{x^2}\right) dx = 1 + \int \left(\frac{1}{x}\right) dx.$$

Subtracting

$$\int \left(\frac{1}{x}\right) dx$$

from both sides of this equality, we obtain $0 = 1$. What went wrong?

If we add explicit bounds of integration to what's written above...

$$\int_a^b \left(\frac{1}{x}\right) dx = \left[\left(\frac{1}{x}\right) x\right]_{x=a}^{x=b} - \int_a^b x \left(-\frac{1}{x^2}\right) dx = \Big[1 - 1\Big] + \int_a^b \left(\frac{1}{x}\right) dx = \int_a^b \left(\frac{1}{x}\right) dx$$

... then it no longer seems unreasonable.

6. Compute the arclength of the graph of $f(x) = \cosh x$ between $x = 0$ and $x = 1$.

$$\int_0^1 \sqrt{1 + \left(\frac{df}{dx}\right)^2}\, dx = \int_0^1 \sqrt{1 + (\sinh x)^2}\, dx = \int_0^1 \cosh x\, dx = \Big[\sinh x\Big]_{x=0}^{x=1} = \sinh(1).$$