Linear Algebra

1. (a) Suppose that $Q_1$ and $Q_2$ are two $n \times n$ orthogonal matrices. Show that the product $Q_1Q_2$ is also an orthogonal matrix.

(b) Consider the matrix

$$
A = \begin{bmatrix}
2 & -1 & 2 \\
1 & 1 & -1 \\
6 & 0 & 2
\end{bmatrix}.
$$

Determine the nullspace of $A$.

2. Consider the system of equations

\begin{align*}
x_1 - x_2 + x_3 + 2x_4 &= 1, \\
-x_1 + 2x_2 + x_3 - x_4 &= 0, \\
2x_1 - x_2 - x_3 + 2x_4 &= 1, \\
11x_1 + x_2 + x_3 - x_4 &= 2, \\
3x_1 + x_2 + 4x_3 + 5x_4 &= 2.
\end{align*}

Write the above in the form $Ax = b$. Using the Gaussian elimination technique to the Augmented matrix $[A|b]$, determine the rank of the matrix $A$. Is there a solution to $x$? If the answer is yes, obtain the solution. If the answer is no, explain.
1. (a) Consider an $n \times n$ elementary unit lower-triangular matrix $L_i$. (Note: A unit lower triangular matrix is a lower triangular matrix with each of the diagonal elements equal to unity. An elementary lower triangular matrix $L_i$ is a lower triangular matrix with possible non-zero elements below the diagonal element in the $i^{th}$ column.)

Let $E_{ij}$ denote the permutation matrix obtained by interchanging the rows $i$ and $j$ of the identity matrix $I$ of size $n$.

Show that the product $E_{35}L_1E_{35}^T$ interchanges only the third and fifth elements in the first column of $L_1$.

(b) Show that permutation matrices are orthogonal.

2. (a) A real symmetric matrix $A$ is said to be positive definite if $x^T A x > 0$ for all $x \neq 0$ ($x$ is an $n$-dimensional vector). Consider the following matrix obtained during the discretization of a differential equation:

$$
A = \begin{bmatrix}
2 & -1 \\
-1 & 2 & -1 \\
& -1 & \ddots & \ddots \\
& & \ddots & \ddots & -1 \\
& & & -1 & 2
\end{bmatrix}
$$

Show that $A$ is positive definite.

(b) Show that the diagonal elements of a positive definite matrix are positive.
Complex variables

1. Let \( u(x, y) \) and \( v(x, y) \) be real-valued functions of the real variables \( x \) and \( y \).

   Also let \( z = x + iy \) and let \( w = u(x, y) + iv(x, y) \).

   a) Sketch a straight line going from \(-1 - i\) to \(2 + i\) on an Argand diagram. This straight line is represented by the variable \( z \).

   b) Sketch the corresponding line for the complex variable \( w = u(x, y) + iv(x, y) \) on an Argand diagram for the transformation \( w = 3iz + 1 \).

   c) As for b), sketch on an Argand diagram the corresponding line for the transformation \( w = \frac{1}{z} \).

2. With \( w \) and \( z \) defined in question 1,

   a) State the Cauchy Riemann condition for \( u(x, y) \) and \( v(x, y) \) to be conjugate functions so that \( w \) is an analytic function of \( z \).

   a) A real-valued function of two real variables is said to be harmonic if it satisfies the two-dimensional version of Laplace’s equation. Is \( v(x, y) = e^x \sin(y) \) harmonic? If so, find the conjugate harmonic function \( u(x, y) \) and express \( w \) directly as a function of \( z \).

   b) Is \( u = x^2 - 2y \) harmonic? If so, find the conjugate harmonic function \( v(x, y) \) and express \( w \) directly as a function of \( z \).
Complex variables

In general let \( w = u + iv \) and \( z = x + iy \) where \( i = \sqrt{-1} \).

1. Given that the above complex numbers can be represented on an Argand diagram, sketch the three straight line paths traced out in the \( z \) plane starting at the origin, A, proceeding to point B at 1, then to point C located at \( i \) before closing at point A. Show these three path segments and use arrows to indicate path directions.

   State the equations for determining points in \( x \) and \( y \) for each of these three path segments.

   Determine and sketch the corresponding path in the \( w \) plane where \( w = z^2 \).

2. If \( w = \sinh(z) \) show that \( u \) and \( v \) satisfy the Cauchy-Riemann condition for differentiability as well as the Laplace equation.

   Derive a general expression for \( \ln(z) \) and hence determine solutions of
   a) \( \ln(-4) \).
   b) \( \ln(3 - 4i) \).